The Higher-Spin/CFT Duality:
The AdS_4/CFT_3 Case

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Vasiliev's theories
- General features and mathematical instruments
- Master fields and equations of motion
- Basic results about Vasiliev's theories

Parity-preserving dualities
- CFT duals of A-type models
- CFT duals of B-type models and checks of the dualities

Parity-violating dualities
- Non-supersymmetric cases
- Few words on supersymmetric extensions

General features

- **Vasiliev's systems** (VS): sets of *classical* non-linear gauge invariant equations for an infinite tower of *interacting* higher spin gauge fields.
  Interactions are highly constrained by higher spin (hs) symmetry
- It does not exist (until now) an action for these set of equations
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AdS/CFT perspective:

- If quantum extensions of VS make sense, they are very interesting in the spirit of AdS / CFT
- General feature: CFT duals of VS are **vector models**

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How hs fields arise

Free conformal theories have a large set of **global symmetries**
$\rightarrow$ Conserved currents of **arbitrary** spin
$\rightarrow$ the gravitational dual must include hs **gauge** fields

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- A vector model (restricted to the *singlet* sector) has precisely the right spectrum to be dual to a **pure** hs gauge theory!

Mathematical instruments

Twistor space

- Beyond coordinates x_μ, we introduce a **twistor space** (Y, Z) $\rightarrow$ coordinates are *commuting*, *spinorial* variables

$$(Y, Z) = (y^\alpha, \bar{y}^{\dot{\alpha}}, z^\alpha, \bar{z}^{\dot{\alpha}}) \quad \alpha = 1, 2 \quad \dot{\alpha} = 1, 2$$

- Lorentzian signature: (y, z) and $(\bar{y}, \bar{z})$ are complex conjugates

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Star product

- It is a *non-commutative* product on the twistor space

$$F(Y,Z) * G(Y,Z) = F \exp\left[\epsilon^{\alpha\beta}(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha})(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta})\right] G$$

- **Identities**: $[y^\alpha, y^\beta]_* = -[z^\alpha, z^\beta]_* = 2\,\epsilon^{\alpha\beta} \quad [y^\alpha, z^\beta]_* = 0$
- **Kleinians**: $K = e^{y^\alpha z_\alpha} \quad \bar{K} = e^{\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}}$

Master fields

- VS are formulated in terms of **Master fields**:
  A 1-form in x^μ-space $W = W_\mu dx^\mu$, a 1-form in Z-space $S = S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}$ and a scalar B, all depend on (Y, Z)

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- For future convenience: it is useful define

$$\hat{\mathcal{A}} = \mathcal{A} + \frac{1}{2} z_\alpha dz^\alpha + \frac{1}{2} \bar{z}_{\dot\alpha} d\bar{z}^{\dot\alpha}$$

Equations of motion

- The equation of motion of VS takes the form

$$d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} = f_*(B * K)dz^2 + \bar{f}_*(B * \bar{K})d\bar{z}^2$$

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- Also a (twisted) Bianchi identity for the field B

$$d_x B + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) = 0$$

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- **Gauge invariance**

$$\delta\hat{\mathcal{A}} = d_x\epsilon + [\hat{\mathcal{A}}, \epsilon]_* \quad \delta B = -\epsilon * B + B * \pi(\epsilon)$$

  $\hat{\mathcal{A}}$ transforms as a connection, B in the twisted adjoint

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Parity preserving/parity violating

- By a field redefinition $f(X)$ can be put in the form $f(X) = X\exp(i\theta(X))$ where $\theta(X) = \theta_0 + \theta_2 X^2 + \theta_4 X^4 + \dots$
  A choice of θ characterizes the interactions in the theory.
- **Parity preserving**: imposing that the theory has a parity symmetry ($x \rightarrow -x$, $y \rightarrow \bar{y}$ and $z \rightarrow \bar{z}$)
  two inequivalent choices
  - $\theta = 0 \rightarrow f(X) = X$ (**A-type**)
  - $\theta = \pi/2 \rightarrow f(X) = i\,X$ (**B-type**)
- **Parity violating**: θ is not constrained

Higher-spin fields in AdS_4

- Vacuum solution of VS: $W = W_0(x, Y)$, $S = 0$, $B = 0$ where

$$W_0 = (e_0)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta} y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}$$

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- Linearized equation: one introduces perturbations

$$W = W_0(x, Y) + \hat{W}(x, Y, Z) \quad S = S(x, Y, Z) \quad B = B(x, Y, Z)$$

  and solves VS at linearized level
- It turns out a tower of **free** higher-spin gauge fields propagating in AdS_4, plus a scalar ϕ of mass $m^2 = -2$ (Breitenlohner and Freedman bound $m^2 > -\frac{9}{4}$)

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- **Interactions**: taken into account by higher order terms in VS

Boundary conditions for the bulk scalar

- We saw that VS has a scalar field ϕ of $m^2 = -2$.
  $\rightarrow$ the dual operator has

$$\Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2} = 2 \text{ or } 1$$

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- Two possible dualities, depending on boundary conditions

$$\phi \sim \phi_0 \, z^2 + \phi_1 \, z + \dots$$

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  Imposing $\phi_0 = 0$ the dual operator has $\Delta = 2$
- If $\Delta = 1$ VS is dual to a **free** O(N) (singlet) vector model
  If $\Delta = 2$ VS is dual to a **critical** O(N) (singlet) vector model

Duality with $\Delta = 1$

- Klebanov and Polyakov conjectured that minimal-VS of A-type with $\Delta = 1$ is dual to a **free**, bosonic O(N) vector model

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- The action for this model is

$$S = \frac{1}{2} \int d^3x \sum_{a=1}^{N} (\partial_\mu \phi^a)^2$$

- The theory has a class of **conserved**, singlet currents

$$J_s = \phi^a \partial_{(\mu_1} \ldots \partial_{\mu_s)} \phi_a \qquad \Delta(J_s) = s + 1$$

there exists one current for every integer even spin s, plus $J_0 = \phi^a \phi_a$ which is dual to the bulk scalar ($\Delta(J_0) = 1$)

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- **AdS/CFT recipe**: Conserved currents are dual to gauge fields
  $\rightarrow$ We need an hs gauge field for every integer even spin
  $\rightarrow$ minimal VS of A-type ($\Delta = 1$) can be the dual!!

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- Also in this case we have the currents

$$J_s = \phi^a\partial_{(\mu_1}\dots\partial_{\mu_s)}\phi_a \qquad J_0 = \phi^a\phi_a$$

they acquire anomalous dimensions when N is finite

$$\Delta(J_s) = s + 1 + \mathcal{O}(\frac{1}{N}) \qquad \Delta(J_0) = 2 + \mathcal{O}(\frac{1}{N})$$

Consequences of the anomalous dimensions

- We have seen that, in the $\Delta = 2$, hs currents acquire an anomalous dimension when N is finite
- Correspondingly these currents are not conserved when N is finite
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- Notice that the interaction term goes as $\frac{1}{N}$: the same behaviour which has the breaking of hs symmetry

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- On the VS side: one can show that hs fields with $s > 2$ acquire a mass (**Higgs mechanism**). This mechanism is intrinsically one-loop and can occur only when we choose $\Delta = 2$

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  → $\Delta = 2$ is dual to a free CFT of fermions
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  J_0 is dual to the parity odd bulk scalar
- $\Delta = 1$ is dual to the interacting fixed point of the same theory, perturbed by a quartic interaction.

Testing the conjectures

- **Trivial test**: In free models, we have seen that the spectrum of conserved currents on the boundary perfectly matches wit the spectrum of hs gauge fields in the bulk
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- Moreover, we have seen that, for interacting models, currents on the boundary are not conserved $\rightarrow$ correspondingly hs fields acquire a mass
- **Non-trivial tests**: at the interacting level, various 3-point functions $< J_{s_1}(x_1)\, J_{s_2}(x_2)\, J_{s_3}(x_3) >$ were computed using Vasiliev's theory, and compared with the corresponding 3-point functions on the boundary
- The duality with critical $O(N)$ model in the $\Delta = 2$ case follows from the duality with the free $O(N)$ model in the $\Delta = 1$ case order by order in $1/N$.

Parity-violating VS and Chern-Simons Vector models

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- We do not want a dynamical gauge field
  $\rightarrow$ parity-preserving CFT can be obtained as the $k \rightarrow \infty$-limit of the $O(N)$-Chern-Simons gauge theories (k is the level)

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Parity-violating duals

It is natural to guess that parity-violating duals are $O(N)$-Chern-Simons gauge theories at finite k

Relations between θ and Chern-Simons level

- We conjectured that parity-violating VS are dual to Chern-Simons vector models (at finite k)
  $\rightarrow$ It should be exists a relation between the phase $\theta(X)$ of VS and the Chern-Simons level k

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for Chern-Simons scalar, and

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- The relations between $\theta_2, \theta_4 \ldots$ and λ are not yet known

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    - A deformation of an A-type VS (parameter λ)
    - A deformation of a B-type VS (parameter $\tilde{\lambda} = 1 - \lambda$)

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  - A deformation of an A-type VS (parameter λ)
  - A deformation of a B-type VS (parameter $\tilde{\lambda} = 1 - \lambda$)
- This suggests that, the same VS has two possible duals: a scalar Chern-Simons vector model (with λ) or a fermionic Chern-Simons vector model (with $\tilde{\lambda}$).

3D-bosonization?

- Suppose that the coefficients θ_2, $\theta_4 \ldots$ simply vanish $\rightarrow$ the relations just found between θ_0 and λ are also the relations between θ and λ
- If this is the case, a generic parity-violating VS with $f(X) = \frac{\pi}{2}\lambda$ can be seen in two ways
  - A deformation of an A-type VS (parameter λ)
  - A deformation of a B-type VS (parameter $\tilde{\lambda} = 1 - \lambda$)
- This suggests that, the same VS has two possible duals: a scalar Chern-Simons vector model (with λ) or a fermionic Chern-Simons vector model (with $\tilde{\lambda}$).
- We should have therefore a phenomenon of **bosonization**

Supersymmetric extensions of VS (1)

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Chan-Paton in VS

- We introduce, as in string-field-theory, a Chan-Paton $U(M)$ factor in VS $\rightarrow$ simply all master fields become $M \times M$ matrices transforming in the adjoint of the gauge group
- **Natural conjecture**: the CFT duals of Chan-Paton VS are the same theories with a new **global** flavour-symmetry
- **Example**: Dual of A-type
  NM free massless scalar

$$\phi_{ai} \quad a = 1 \dots N \quad i = 1 \dots M$$

  restricted to the $O(N)$ singlet sector
- hs currents valued in the adjoint of the global flavour group

Supersymmetric extensions of VS (2)

- To introduce SUSY: we introduce Grassmannian variables $\psi_i \quad i = 1 \ldots n$ with Clifford Algebra $\{\psi_i, \psi_j\} = 2\delta_{ij}$
- All the others twistorial variables commute with ψ_i
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- All the others twistorial variables commute with ψ_i
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- **Parity operator** R: it becomes

$$R = K\bar{K}\Gamma \qquad \Gamma = i^{\frac{n(n-1)}{2}}\psi_1\psi_2\ldots\psi_n$$

- Truncation conditions are the same (R new)

$$[R, W]_* = \{R, S\}_* = [R, B]_* = 0$$

with this modification one can show that now the theory has bulk spinorial fields (odd functions of ψ_i) and they are fermions

Boundary conditions break SUSY

- **Non supersymmetric models**:
  We saw that hs symmetry is encoded in the gauge variations

  $$\delta\hat{\mathcal{A}} = d_x\epsilon + [\hat{\mathcal{A}}, \epsilon]_* \quad \delta B = -\epsilon * B + B * \pi(\epsilon)$$

  where the gauge parameter ϵ is a function of the x^μ and of the twistorial variables (Y, Z)
- Moreover, we saw that hs symmetry is in general break by boundary conditions imposed on the scalar field ϕ

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- VS which preserve $\mathcal{N} = 0, 1, 2, 3, 4, 6$ were found!!

The ABJ triality (1)

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- It was conjectured that $n = 6$, parity-violating VS with $\mathcal{N} = 6$ boundary conditions and $U(M)$ Chan-Paton factor is dual to $U(N)_k \times U(M)_{-k}$ ABJ theory in the limit

$$N \to \infty \quad k \to \infty \quad M \,\text{finite}$$

N and M are treated on different footing
$\to$ because $U(N)$ is the original gauge symmetry, $U(M)$ obtained gauging the flavour (Chan-Paton) symmetry

The ABJ triality (2)

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$$N \to \infty \quad M \to \infty \quad k << N << k^5$$

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Interpretation of the bulk to bulk duality

When λ_{bulk} grows, the interactions between hs fields become stronger. This leads to bound states of hs fields, these bound states are the **strings** of Type IIA theory